Characterization of homomorphism functions

László Lovász

2013

1 Statement of the theorem

Graph means simple graph with loops allowed (i.e., a symmetric binary relation).

An *arrow* is a homomorphism between two graphs. Every arrow $\varphi : F \to G$ has a *tail* (the graph F), and a *head* (the graph G). We say that arrows $\alpha : F \to G$ and $\beta : F \to H$ are *equivalent*, in notation $\alpha \simeq \beta$, if there is an isomorphism $\xi : G \to H$ for which $\alpha\xi = \beta$ (note: we write the composition of two homomorphisms in the order they should be performed). We denote by $\mathrm{Hom}_F^{\mathrm{out}}$ the set of arrows with tail in F.

The *pushout* of two arrows $\alpha : F \to G$ and $\beta : F \to H$ is defined as follows: we take the disjoint union of G and H, and we identify $\alpha(v)$ and $\beta(v)$ for every node $v \in V(F)$. (Since α and β are not necessarily injective, this may result in identifying more than two nodes). This gives a graph L, along with homomorphisms $\alpha' : G \to L$ and $\beta' : H \to L$ such that $\alpha\alpha' = \beta\beta'$. We will denote β' by $\beta^\top\alpha$ and α' by $\alpha^\top\beta$. The arrow $\alpha\alpha' = \beta\beta'$ will be denoted by $\alpha \wedge \beta$.

The following fact is easy to verify:

Lemma 1.1 *$\alpha, \beta \in \mathrm{Hom}_F^{\mathrm{out}}$, and $L = h(\alpha\wedge\beta)$. Then for any two arrows $\xi : F \to J$ and $\zeta : G \to J$ such that $\alpha\xi = \beta\zeta$ there exists a unique arrow $\eta : L \to J$ such that $\alpha'\eta = \xi$ and $\beta'\eta = \zeta$.*

We take formal linear combinations of arrows in $\mathrm{Hom}_F^{\mathrm{out}}$ (for a graph F), to get the linear space $\mathcal{Q}_F$. This will be infinite dimensional, but we will see that it has interesting finite dimensional factors.

Let f be a graph parameter (a real valued function defined on graphs, invariant under isomorphism). We say that f is *multiplicative*, if $f(F\dot\cup G) = f(F)f(G)$ for any two graphs F and G.

For every graph F, we define a (possibly infinite) symmetric matrix $M(f, F)$, whose rows and columns are indexed by arrows in $\mathrm{Hom}_F^{\mathrm{out}}$, and whose entry in row α and column β is $f(h(\alpha \wedge \beta))$ (since $\alpha \wedge \beta$ is determined up to isomorphism, this is well defined).

Theorem 1.2 *Let f be a graph parameter. There is a graph G such that $f = |\mathrm{Hom}(., G)|$ if and only if the following conditions are fulfilled:*

(F1) $f(\emptyset) = 1$,

(F2) *f is multiplicative, and*

(F3) *$M(f, F)$ is positive semidefinite for every graph F.*

We note that if there is an epimorphism from F to G, then $M(f,G)$ is a submatrix of $M(f,F)$. Thus it would be enough to require the semidefiniteness condition for edgeless graphs.

Corollary 1.3 *Conditions* (F1)–(F3) *of the theorem imply that the values of f are non-negative integers, and the rank of $M(f,F)$ is finite for every F.*

We can prove the "only if" part of Theorem 1.2 right away; the "if" direction will take most of the work.

Proof. **(The easy direction.)** Suppose that $f = \mathsf{hom}(.,G)$ for some graph G. Then $f(z) = 1$ for the zero graph t, and f is multiplicative by the definition of direct product. To show that $M(f,F)$ is positive semidefinite, consider any $\gamma : F \to H$ and $\delta : F \to L$, and let $u = h(\gamma \wedge \delta)$. Note that, by the definition of pushouts, $f(u)$ is the number of pairs of arrows (φ, ψ) $(\varphi : H \to G,\ \psi : L \to G)$ such that $\gamma\varphi = \delta\psi$. Fix any arrow $\mu : F \to G$, and let M_γ^μ denote the number of arrows $\varphi : H \to G$ such that $\gamma\varphi = \mu$. Clearly $M(f,F)_{\gamma,\delta} = \sum_\mu M_\gamma^\mu M_\delta^\mu$, and so the matrix $M(f,F)$ is the sum $|\mathrm{Hom}(F,G)|$ positive semidefinite matrices of rank 1. □

2 Algebras of arrows

For each graph F, the operation $\wedge$ defines a semigroup on $\mathrm{Hom}_F^{\mathrm{out}}$. This extends to $\mathcal{Q}_F$ by distributivity.

If $\varphi : F \to G$ is any arrow, then $\alpha \mapsto \varphi\alpha$ extends to a linear map $\mathcal{Q}_G \to \mathcal{Q}_F$, which we denote by $x \mapsto \varphi x$. The map $\beta \mapsto \varphi^\top\beta$ extends to a linear map $\mathcal{Q}_F \to \mathcal{Q}_G$, which we denote by $x \mapsto \varphi^\top x$.

Let f be a graph parameter. It will be convenient to extend it to arrows, and define $f(\varphi) = f(h(\varphi))$. Clearly, this extension is invariant under isomorphism of arrows. We can extend f to the algebras $\mathcal{Q}_F$ linearly. It follows from the definition that for $x \in \mathcal{Q}_G$ and $\varphi : F \to G$ we have $f(\varphi x) = f(x)$.

For $\alpha, \beta \in \mathrm{Hom}_F^{\mathrm{out}}$, we define $\langle \alpha, \beta \rangle = f(\alpha \wedge \beta)$, which yields a (generally indefinite) inner product on $\mathcal{Q}_F$.

To study this inner product, we need some simple identities. In the special case when the involved quantum arrows are ordinary arrows, these identities can be verified directly. For the general case, they follow by linearity.

$$(\varphi\psi)^\top x = \psi^\top(\varphi^\top x) \qquad (x \in \mathcal{Q}_F,\ \varphi : F \to G,\ \psi : G \to H,); \tag{1}$$

$$\langle x, \varphi y \rangle = \langle \varphi^\top x, y \rangle \qquad (x \in \mathcal{Q}_F,\ \mathrm{y}\in \mathcal{Q}_G,\ \varphi : F \to G) \tag{2}$$

(these identities justify the notation $\varphi^\top$);

$$\varphi^\top(x \wedge y) = \varphi^\top x \wedge \varphi^\top y \qquad (x, y \in \mathcal{Q}_F,\ \varphi : F \to G). \tag{3}$$

So the map $\varphi^\top$ preserves this product. This is not true for φ in general, but it is true in an important special case:

$$\varphi(x \wedge y) = \varphi x \wedge \varphi y \qquad (x, y \in \mathcal{Q}_G,\ \varphi : F \to G \text{ epic}). \tag{4}$$

Using this, it is easy to see that an epimorphism preserves inner product:

$$\langle \varphi x, \varphi y\rangle = \langle x, y\rangle \qquad (x, y \in \mathcal{Q}_G,\ \varphi : F \to G \text{ epic}). \tag{5}$$

The following identity is called the *Frobenius identity*:

$$\langle x \wedge y, z\rangle = \langle x, y \wedge z\rangle \qquad (x, y, z \in \mathcal{Q}_F). \tag{6}$$

Finally, we need the following:

$$\langle (\varphi_2^{\mathsf{T}}\varphi_1)^{\mathsf{T}} x_1, (\varphi_1^{\mathsf{T}}\varphi_2)^{\mathsf{T}} x_2\rangle = \langle \varphi_1 x_1, \varphi_2 x_2\rangle \qquad (x_i \in \mathcal{Q}_{G_i},\ \varphi_i : F \to G_i). \tag{7}$$

Let

$$\mathcal{N}_F = \{x \in \mathcal{Q}_F : \langle x, y\rangle = 0 \text{ for all } y \in \mathcal{Q}_F\},$$

then $\mathcal{N}_F$ is an ideal in the algebra $\mathcal{Q}_F$, since if $x \in \mathcal{N}_F$, then by (6), we have for all $y, z \in \mathcal{Q}_F$, $\langle x \wedge y, z\rangle = \langle x, y \wedge z\rangle = 0$, and hence $x \wedge y \in \mathcal{N}_F$. It is also clear that if $\alpha \simeq \beta$, then $\alpha - \beta \in \mathcal{N}_F$.

We can form the factor $\mathcal{A}_F = \mathcal{Q}_F/\mathcal{N}_F$. This is another associative and commutative algebra with an inner product. The coset of id_F is an identity element in $\mathcal{A}_F$.

While the algebra $\mathcal{Q}_F$ is infinite dimensional, the factor algebra $\mathcal{A}_F$ is much smaller. We are going to show that it is finite dimensional, and it can even consist of the 0 element only. For example, if $f = \mathsf{hom}(., H)$ for some graph H, and F has no homomorphism into H, then $\mathsf{hom}(F, H) = 0$ and more generally, $\mathsf{hom}(G, H) = 0$ for every G for which there exists a homomorphism $F \to G$. Hence $\langle \varphi, \psi\rangle = f(h(\varphi \wedge \psi)) = 0$ for any two arrows $\varphi, \psi \in \mathrm{Hom}_F^{\mathrm{out}}$, and so $N_F = \mathcal{Q}_F$. In this case, $\mathrm{id}_F \equiv 0$.

We need a lemma relating algebras sitting at different graphs.

Lemma 2.1 *Let F and G be two graphs and $\varphi : F \to G$.*

(a) *If $x \in \mathcal{N}_G$ then $\varphi x \in \mathcal{N}_F$. The map $x \mapsto \varphi x$ induces a linear map $\mathcal{A}_G \to \mathcal{A}_F$.*

(b) *If $y \in \mathcal{N}_F$ then $\varphi^\top y \in \mathcal{N}_G$. The map $y \mapsto \varphi^\top y$ induces an algebra homomorphism $\mathcal{A}_F \to \mathcal{A}_G$.*

(c) *If φ is an epimorphism, then $\varphi x \in \mathcal{N}_F$ implies that $x \in \mathcal{N}_G$. The map $x \mapsto \varphi x$ induces an injective algebra homomorphism $\mathcal{A}_G \to \mathcal{A}_F$.*

Proof. (a) To prove that $\varphi x \in \mathcal{N}_F$, we want to prove that $\langle \varphi x, y\rangle = 0$ for all $y \in \mathcal{Q}_F$. By (2), $\langle \varphi x, y\rangle = \langle x, \varphi^\top y\rangle$, which is 0 as $x \in \mathcal{N}_G$. The second assertion follows from this trivially.

(b) To prove that $\varphi^\top y \in \mathcal{N}_G$, we want to prove that $\langle \varphi^\top y, x\rangle = 0$ for all $x \in \mathcal{Q}_G$. Similarly as before, $\langle \varphi^\top y, x\rangle = \langle y, \varphi x\rangle = 0$ as $y \in \mathcal{N}_F$. The second assertion follows from this and (3).

(c) Assume that $\varphi x \in \mathcal{N}_F$ for some $x \in \mathcal{Q}_G$. Then $\langle \varphi x, y\rangle = 0$ for every $y \in \mathcal{Q}_F$, in particular, $\langle \varphi x, \varphi z\rangle = 0$ for every $z \in \mathcal{Q}_G$. Then by (5), $\langle x, z\rangle = 0$ for every $z \in \mathcal{Q}_G$, and so $x \in \mathcal{N}_G$. The second assertion follows from this, using (a) and (4). □

This Lemma implies that identities (1)–(7) make sense and remain valid for x, y, z in the factor algebra $\mathcal{A}_F$.

3 Semidefiniteness

To use the hypothesis about semidefiniteness, we start with a simple observation:

Lemma 3.1 *The inner product $\langle .,. \rangle$ is positive semidefinite on $\mathcal{Q}_F$ if and only if the matrix $M(f,F)$ is positive semidefinite.*

Proof. Let $x = \sum_\alpha x_\alpha \alpha \in \mathcal{Q}_F$. We can also think of x as a column vector indexed by arrows $\alpha \in \mathrm{Hom}_F^{\mathrm{out}}$. Then

$$\begin{aligned}\langle x, x\rangle &= \sum_{\alpha,\beta} \langle \alpha, \beta\rangle x_\alpha x_\beta = \sum_{\alpha,\beta} f(\alpha \wedge \beta) x_\alpha x_\beta \\ &= \sum_{\alpha,\beta} M(f,F)_{\alpha,\beta} x_\alpha x_\beta = x^{\mathsf{T}} M(f,F) x.\end{aligned}$$

This is nonnegative for all $x \in \mathcal{Q}_F$ if and only if $M(f,F)$ is positive semidefinite. □

From now on we assume that all of the matrices $M(f,F)$ are positive semidefinite, and so the inner product $\langle .,. \rangle$ is positive semidefinite on every $\mathcal{Q}_F$ and so it is positive definite on $\mathcal{A}_F$.

To warm up, let us mention an easy consequence of positive semidefiniteness. (The proof is an exercise.)

Lemma 3.2 *If F is a spanning subgraph of G, then $f(F) \ge f(G)$.*

The following is the first substantial lemma.

Lemma 3.3 *The algebra $\mathcal{A}_F$ is finite dimensional and* $\dim(\mathcal{A}_F) \le f(F)$.

Proof. Let $\eta_1, \eta_2 : F \to F \oplus F$ be the canonical embeddings. There is a unique arrow $\varphi : F \oplus F \to F$ such that $\eta_1 \varphi = \eta_2 \varphi = \mathrm{id}_F$.

Let $e_1, \dots, e_n$ be mutually orthogonal unit vectors in $\mathcal{A}_F$. Both assertions will follow if we prove that $n \le f(F)$.

Let

$$x = \sum_{i=1}^n (\eta_1^\top e_i \wedge \eta_2^\top e_i) - \varphi.$$

Then

$$\begin{aligned}\langle x, x\rangle = &\sum_{i=1}^N \langle \eta_1^\top e_i \wedge \eta_2^\top e_i,\ \eta_1^\top e_i \wedge \eta_2^\top e_i\rangle + 2\sum_{i<j} \langle \eta_1^\top e_i \wedge \eta_2^\top e_i,\ \eta_1^\top e_j \wedge \eta_2^\top e_j\rangle \\ &- 2\sum_{i=1}^n \langle \eta_1^\top e_j \wedge \eta_2^\top e_j, \varphi\rangle + \langle \varphi, \varphi\rangle. \qquad (8)\end{aligned}$$

Here using (7),

$$\langle \eta_1^\top e_i \wedge \eta_2^\top e_i,\ \eta_1^\top e_i \wedge \eta_2^\top e_i\rangle = f(e_i \wedge e_i)^2 = \langle e_i, e_i\rangle^2 = 1.$$

Similarly,

$$\langle \eta_1^\top e_i \wedge \eta_2^\top e_i,\ \eta_1^\top e_j \wedge \eta_2^\top e_j \rangle = \langle e_i, e_j \rangle^2 = 0.$$

Furthermore, using that $\eta_2^\top e_i \wedge \varphi = \varphi(\varphi^\top \eta_2^\top e_i) = \varphi((\eta_2\varphi)^\top e_i) = \varphi e_i$, we have

$$\langle \eta_1^\top e_i \wedge \eta_2^\top e_i, \varphi \rangle = \langle \eta_1^\top e_i, \eta_2^\top e_i \wedge \varphi \rangle = \langle \eta_1^\top e_i, \varphi e_i \rangle = \langle e_i, \eta_1 \varphi e_i \rangle = \langle e_i, e_i \rangle = 1.$$

Since φ is an idempotent in $\mathcal{Q}_{a\oplus a}$,

$$\langle \varphi, \varphi \rangle = f(\varphi \wedge \varphi) = f(\varphi).$$

Hence by (8),

$$\langle x, x \rangle = n + 0 - 2n + f(\varphi) = f(F) - n.$$

Since this is nonnegative, the lemma follows. □

Using that by Lemma 3.2 $f(F) \le f(O_{|V(F)|}) = f(K_1)^{|V(F)|}$, we get:

Corollary 3.4 $\dim(\mathcal{A}_F) \le f(K_1)^{|V(F)|}$.

Lemma 3.5 *The algebra $\mathcal{A}_F$ is isomorphic to $\mathbb{R}^d$, where $d = \dim(\mathcal{A}_F)$.*

Proof. For every $x \in \mathcal{A}_F$, let B_x denote the linear transformation of $\mathcal{A}_F$, defined by $B_x y = x \wedge y$. Clearly $B_x B_y = B_{xy} = B_{yx} = B_y B_x$. Identity (6) implies that these linear transformations are symmetric. By basic linear algebra, this implies that there is a basis in which all these transformations B_x are diagonal. Writing $B_x = \mathrm{diag}(b_x)$, where $b_x \in \mathbb{R}^d$, we get that $x \mapsto b_x$ defines an embedding $\mathcal{A}_F \to \mathbb{R}^d$. Since the dimensions of $\mathcal{A}_F$ and $\mathbb{R}^d$ are equal, it follows that they are isomorphic. □

Using these properties, we can construct one of our main tools. Recall that an algebra element p is an *idempotent*, if $p \wedge p = p$. In $\mathbb{R}^d$, idempotent elements are precisely the 0-1 vectors. Considering the algebra elements corresponding to the standard basis in $\mathbb{R}^d$, we conclude:

Corollary 3.6 *The algebra $\mathcal{A}_F$ has a (unique) orthogonal basis B_F consisting of idempotents.*

We call these idempotents the *basic idempotents* in $\mathcal{A}_F$. Every idempotent in $\mathcal{A}_F$ is the sum of a subset of B_F, and in particular

$$\mathbb{1}_F = \sum_{p \in B_F} p. \tag{9}$$

Next, we compare basic idempotents belonging to two graphs. Let $\varphi : F \to G$. Since $\varphi^\top : \mathcal{A}_F \to \mathcal{A}_G$ is an algebra homomorphism, $\varphi^\top p$ is an idempotent in $\mathcal{A}_G$ for any $p \in B_F$, and so it can be written as a sum of basic idempotents:

$$\varphi^\top p = \sum_{q \in B_{\varphi,p}} q. \tag{10}$$

Furthermore, we have $\varphi^\top \mathbb{1}_G = \mathbb{1}_F$, and so (9) implies that

$$\sum_{p\in B_F} \varphi^\top p = \varphi^\top \mathbb{1}_F = \mathbb{1}_G = \sum_{q\in B_G} q. \tag{11}$$

This implies that the sets $B_{\varphi,p}$ ($p \in B_F$) partition the set B_G, and we have

$$B_{\varphi,p} = \{q \in B_G : \varphi^\top p \wedge q = q\}.$$

Lemma 3.7 *Let $p \in B_F$, $q \in B_G$, and $\varphi : F \to G$. Then $q \in B_{\varphi,p}$ if and only if φq is a scalar multiple of p. If this happens, then*

$$\varphi q = \frac{f(q)}{f(p)} p.$$

If φ is an epimorphism, then $\varphi q = p$.

Note that here $f(q) = f(q \wedge q) = \langle q, q\rangle > 0$ and similarly $f(p) > 0$.

Proof. Let $q \in B_{\varphi,p}$ and set $x = (f(q)/f(p)p$. We claim that $\langle \varphi q, p'\rangle = \langle x, p'\rangle$ for all $p' \in B_F$. If $p' \neq p$, then

$$\langle \varphi q, p'\rangle = \langle q, \varphi^\top p'\rangle = 0 = \langle x, p'\rangle,$$

since $\langle p, p'\rangle = 0$. On the other hand, for $p' = p$ we have

$$\langle \varphi q, p\rangle = \langle q, \varphi^\top p\rangle = f(q \wedge (\varphi^\top p)) = f(q) = \langle x, p\rangle.$$

Hence it follows that $\varphi q = x$.

Suppose that φq is a scalar multiple of p. We know that q belongs to $B_{\varphi,p'}$ for some $p' \in B_G$, hence $q\varphi = \frac{f(q)}{f(p')} p'$, and so $p = p'$.

If φ is epic, then it defines an injective algebra homomorphism from $\mathcal{A}_G$ to $\mathcal{A}_F$ by Lemma 2.1(c), and hence using (4),

$$\frac{f(q)}{f(p)} p = \varphi q = \varphi(q \wedge q) = (\varphi q) \wedge (\varphi q) = \Big(\frac{f(q)}{f(p)} p\Big) \wedge \Big(\frac{f(q)}{f(p)} p\Big) = \Big(\frac{f(q)}{f(p)}\Big)^2 p,$$

which implies that $f(q)/f(p) = 1$. $\square$

4 Simplified idempotents

Let F and G be two graphs and $x \in \mathcal{A}_F$, $y \in \mathcal{A}_G$. We say that y is a *simplification* of x if there exists an epimorphism $\varphi : F \to G$ such that $x = \varphi y$. It is clear that a simplification of a simplification is a simplification. We say that the simplification y of x is *proper*, if φ is not an isomorphism.

Lemma 4.1 *Every $x \in \mathcal{A}_F$ has a unique "simplest" simplification y such that for every other simplification z of x, y is a simplification of z.*

Proof. A proper simplification "sits" at a graph with either fewer number of nodes or the same number of nodes but with more edges. This implies that there is a simplification y of x such that y has no simplification other than itself. We claim that if z is any other simplification of x, then y is a simplification of z.

Let $y \in \mathcal{A}_G$ and $z \in \mathcal{A}_H$, and let $\varphi : F \to G$ and $\psi : F \to H$ be epimorphisms such that $x = \varphi y = \psi z$. Then

$$x = \varphi y = \varphi(\mathbb{1}_G \wedge y) = \varphi \mathbb{1}_G \wedge \varphi y = \varphi \mathbb{1}_G \wedge \psi z.$$

By (7), setting $u = (\varphi^\top \psi)^\top \mathbb{1}_G \wedge (\psi^\top \varphi)^\top z$, we have $x = \varphi \mathbb{1}_G \wedge \psi z = (\varphi \wedge \psi) u$. Since $\varphi \wedge \psi$, $\psi^\top \varphi$ and $\varphi^\top \psi$ are epimorphisms, this implies that u is a simplification of each of x, y and z. So we must have $u = y$, which implies that y is a simplification of z as claimed. □

So it follows that every $x \in \mathcal{A}_F$ has a "most simplified" version, unique up to isomorphism, which we denote by $s(x)$.

Lemma 4.2 *If p is a basic idempotent, then every simplification of p is a basic idempotent.*

Proof. Let $p \in B_F$, $y \in \mathcal{A}_G$ and $p = \varphi y$, where $\varphi : F \to G$ is epic. Write $y = \sum_{q \in B_G} \lambda_q q$. Then $p = \varphi y = \sum_{q \in B_G} \lambda_q \varphi q$. By Lemma 3.7, the algebra elements φq are basic idempotents in $\mathcal{A}_F$, and so one of them must be equal to p. Hence $\varphi q = \varphi y$ for this basic idempotent, and by Lemma 2.1(c), this implies that $y = q$. □

Basic idempotents of the form $s(p)$ will be called *simplified*.

Lemma 4.3 *Let $p \in B_F$ be a simplified basic idempotent, and let $\varphi : F \to G$. Suppose that φ is not a monomorphism. Then $\varphi^\mathsf{T} p = 0$, and there is no $y \in \mathcal{Q}_G$ with $p = \varphi y$.*

Proof. We can write $\varphi = \gamma\delta$, where $\gamma : F \to J$ is an epimorphism and $\delta : J \to F$ is a monomorphism. If $\varphi^\mathsf{T} p = (\gamma\delta)^\mathsf{T} p = \delta^\mathsf{T} \gamma^\mathsf{T} p \neq 0$, then $\gamma^\mathsf{T} p \neq 0$. Then there exists a $q \in B_{J,\gamma}$, and for this idempotent, $p = \gamma q$ by Lemma 3.7. Since p is simplified, this implies that γ is an isomorphism. If $p = \varphi y = \gamma(\delta y)$, then by the assumption that p is simplified, it follows that γ is an isomorphism. In both cases, φ must be a monomorphism. □

Lemma 4.4 *Let $p \in B_F$ be a simplified basic idempotent, $\varphi : F \to G$, $q \in B_{\varphi,p}$ and let $q = \eta s$ for some $s \in \mathcal{A}_H$ and epimorphism $\eta : G \to H$. Then $\varphi\eta$ is a monomorphism.*

Proof. Let $\mu = \varphi\eta$. We know that $s \in B_{\eta,q}$ and $q \in B_{\varphi,p}$, which implies that $s \in B_{\varphi\eta,p}$. Furthermore, $(\varphi\eta)^\mathsf{T} p \neq 0$, since $\langle (\varphi\eta)^\mathsf{T} p, s \rangle = \langle \varphi^T p, \eta s \rangle = \langle \varphi^\mathsf{T} p, q \rangle = 1$. Lemma 4.3 implies that μ is a monomorphism. □

Lemma 4.5 *Let $p \in B_F$ be a simplified basic idempotent, and let $\varphi, \psi : G \to F$ be epimorphisms such that $\varphi p = \psi p$. Then φ and ψ are equal up to an automorphism of F.*

Proof. It can be checked that $z = (\varphi^\top\psi)^\top p \wedge (\psi^\top\varphi)^\top p \in \mathcal{A}_H$ satisfies $\varphi p \wedge \psi p = (\varphi \wedge \psi)z$. On the other hand, $\varphi p = \psi p$ is an idempotent, and so $\varphi p \wedge \psi p = \varphi p$. Thus $\varphi(\varphi^\top\psi)z = (\varphi \wedge \psi)z = \varphi p$, whence $(\varphi^\top\psi)z = p$ as φ is epic. But $\varphi^\top\psi$ is also an epimorphism, and since p is simplified, it follows that it is an isomorphism. Similarly, $\psi^\top\varphi$ is an isomorphism, and hence $\sigma = (\psi^\top\varphi)(\varphi^\top\psi)^{-1}$ is an automorphism of F. Since $\psi\sigma = \varphi$, this proves the lemma. □

Lemma 4.6 *If $p \in B_F$ is a simplified basic idempotent, then for every graph G,*

$$\dim(\mathcal{A}_G) \geq \frac{|\mathrm{Hom}^{\mathrm{epi}}(G,F)|}{|\mathrm{Hom}^{\mathrm{epi}}(F,F)|}.$$

Proof. For every $\varphi \in \mathrm{Hom}^{\mathrm{epi}}(G,F)$, φp is a basic idempotent in $\mathcal{A}_G$. By Lemma 4.5, if $\varphi p = \psi p$, then $\psi = \varphi\sigma$ for some automorphism σ of F. This implies that $\mathcal{A}_G$ has at least $|\mathrm{Hom}^{\mathrm{epi}}(G,F)|/|\mathrm{Hom}^{\mathrm{epi}}(F,F)|$ different basic idempotents. □

Lemma 4.7 *The number of simplified basic idempotents is finite.*

Proof. Let F be a graph on k nodes such that $\mathcal{A}_F$ has a simplified basic idempotent p, and let $G = O_n$ be the edgeless graph on n nodes. Then $|\mathrm{Hom}^{\mathrm{epi}}(G,F)| \geq m^{m-k}$, and so combining Corollary 3.4 with Lemma 4.6, we get that

$$m^{k-m} \leq \dim(\mathcal{A}_G) \leq f(K_1)^k.$$

Letting $k \to \infty$, we get $m \leq f(K_1)$. □

5 Conclusion

Lemma 5.1 *Let $p \in B_F$ be a simplified basic idempotent where F is has maximum number of edges and let $\varphi : F \to G$. Then*

$$\varphi^\top p = \sum_{\psi:\, \varphi\psi = \mathrm{id}_F} \psi p.$$

Proof. If φ is not monomorphic, then both sides are 0. We want to show that the right side contains exactly the same terms as the representation

$$\varphi^\top p = \sum_{q \in B_{\varphi,p}} q.$$

Every ψ to be considered in the sum is epic, and hence ψp is an idempotent in $\mathcal{A}_G$. Lemma 3.7 implies that ψp is basic, and since $\varphi(\psi p) = p$, it also implies that $\psi p \in B_{\varphi,p}$.

Conversely, let $q \in B_{\varphi,p}$; we want to prove that $q = \psi p$ for some map ψ with $\varphi\psi = \mathrm{id}_F$. Let $s(q) \in B_H$ and $q = \eta s$ for some epimorphism $\eta : G \to H$. By Lemma 4.4, $\mu = \varphi\eta$ is a monomorphism from H into F. By the maximality of p, this implies that μ is an isomorphism, and so $p = \mu s(q)$. The map $\psi = \eta\mu^{-1}$ satisfies $\varphi\psi = \mathrm{id}_F$ and $\psi p = \eta\mu^{-1}p = \eta s = q$.

Finally, we note that different maps ψ give different terms by Lemma 4.5. □

Lemma 5.2 *For any two graphs F, G and simplified basic idempotent $p \in \mathcal{A}_F$ such that F has a maximum number of edges,*

$$\sum_{\varphi\colon G\to F} \varphi p = f(p)\mathbb{1}_G. \tag{12}$$

Proof. Let $H = F\dot{\cup}G$, and let γ and δ be the canonical embeddings of $F \to H$ and $G \to H$. For each $\varphi : G \to F$, let $\overline{\varphi} : H \to F$ be defined as φ on G and id_F on F. We then have $\gamma\overline{\varphi} = \mathrm{id}_F$ and $\delta\overline{\varphi} = \varphi$. It is easy to see that these two conditions determine $\overline{\varphi}$. Hence, with Lemma 5.1,

$$\sum_{\varphi\colon G\to F} \varphi p = \sum_{\psi\colon \gamma\psi=\mathrm{id}_F} \delta\psi p = \delta\Big(\sum_{\psi\colon \gamma\psi=\mathrm{id}_F} \psi p\Big) = \delta(\gamma^\top p).$$

By (2), we have for each $y \in \mathcal{A}_G$:

$$\langle y, \delta\gamma^\top p\rangle = \langle \delta^\top y, \gamma^\top p\rangle = \langle \gamma\delta^\top y, p\rangle = f(y)f(p) = \langle y, f(p)\mathbb{1}_G\rangle.$$

This implies that $\delta(\gamma^\top p) = f(p)\mathbb{1}_G$. □

We are now ready to prove our main theorem.

Proof. Let $p \in \mathcal{A}_F$ be a simplified basic idempotent with maximum number of edges. Then for every graph G, by Lemma 5.2,

$$f(G) = f(\mathbb{1}_G) = \frac{1}{f(p)} \sum_{\varphi\colon G\to F} f(\varphi p) = |\mathrm{Hom}(G, F)|.$$

This completes the proof. □