Selected topics in graph theory

Homework # 1

Date: April 13, 2023

Due: April 30, 2023.

(Solutions must be submitted electronically. I recommend pdf compiled from latex.)

1.1 Problem. Prove that for any two simple graphs F and G,

$$|t_{\text{inj}}(F,G) - t(F,G)| \le \frac{1}{|V(G)|}\binom{|V(F)|}{2}. \tag{1}$$

1.2 Problem. Homomorphisms between directed graphs are defined just like for undirected graphs, but they must also preserve the orientation of the edges.

Let G be a directed graph, let $\overrightarrow{P}_n$ denote the directed path on n nodes, and let $\overrightarrow{K}_n$ denote the complete graph on $\{1,2,\dots,n\}$ in which every edge is oriented from the node with lower label to the node with larger label (transitive tournament). Prove that G has a homomorphism into $\overrightarrow{K}_n$ if and only if $\overrightarrow{P}_{n+1}$ has no homomorphism into G.

1.3 Problem. The *threshold graphs* H_n are graphs defined as follows:

$$V(H_n) = \{1,\dots,n\},$$

$$E(H_n) = \{(i,j) : \; i+j \le n\}.$$

Prove that for every simple graph F, $t(F,H_n)$ has a limit as $n \to \infty$.

1.4 Problem. For a symmetric $n \times n$ matrix A, define the *cut norm* by

$$\|A\|_\square = \max_{S,T\subseteq\{1,\dots,n\}} \left| \sum_{i\in S,\, j\in T} A_{ij} \right|,$$

and a modified cut norm by

$$\|A\|_\blacksquare = \max_{S\subseteq\{1,\dots,n\}} \left| \sum_{i,j\in S} A_{ij} \right|.$$

Prove that

$$\frac{1}{2}\|A\|_\square \le \|A\|_\blacksquare \le \|A\|_\square.$$

1.5 Problem. (Bonus problem) Prove that for every graph G,

$$t(C_6,G)^2 \le t(C_4,G)t(C_8,G).$$

(Here C_m denotes the cycle of length m.)