Selected topics in graph theory

Homework #2

Date: May 3, 2023

Due: May 14, 2023.

(Solutions must be submitted electronically. I recommend pdf compiled from latex. Send it to lovasz@caesar.elte.hu.)

1.1 Problem. For an $n \times b$ real matrix A, define

$$\|A\|_1 = \sum_{i,j} |A_{ij}|.$$

(a) Prove that

$$\|A\|_\square \le \|A\|_1 \le n^2 \|A\|_\square$$

for every $n \times n$ matrix A. (b) Improve the factor n^2 to $2n$. (c) [Difficult, bonus problem!] Improve the factor n^2 to $10\sqrt{n}$.

1.2 Problem. For a graphon W, define the following quantities, analogous to the minimum degree and maximum degree of a graph:

$$\delta(W) = \inf_{x\in[0,1]} \int_0^1 W(x,y)\,dy, \qquad \Delta(W) = \sup_{x\in[0,1]} \int_0^1 W(x,y)\,dy,$$

Prove that for any tree T on k nodes, $\delta(W)^k \le t(T,W) \le \Delta(W)^k$.

1.3 Problem. For a simple graph F on $V(F) = \{1,\dots,k\}$ and graphon W, we define the "induced density"

$$t_{\text{ind}}(F,W) = \int_{[0,1]^k} \prod_{ij\in E(F)} W(x_i,x_j) \prod_{ij\in E(\overline{F})} (1 - W(x_i,x_j))$$

(here $\overline{F}$ is the complementary graph of F). Prove that if $G_n \to W$ for some graph sequence (G_n), then $t_{\text{ind}}(F,G_n) \to t_{\text{ind}}(F,W)$.

1.4 Problem. We generate a randomly growing graph sequence G_n as follows. We start with a single node. At the n-th iteration, a new node is born, and then every pair of nonadjacent nodes is connected with probability 1/n. Prove that with probability 1, this sequence tends to the graphon $W(x,y) = 1 - \max(x,y)$.

1.5 Problem. Starting with a single node, at each step we create a new node, flip a coin, and connect the new node either to all previous nodes, or to none of them, depending on the outcome of the coin flip. Prove that with probability 1, the sequence of graphs we obtain converges to the graphon

$$W(x,y) = \begin{cases} 1, & \text{if } x+y \le 1,\\ 0, & \text{otherwise.}\end{cases}$$