Topological Methods in Combinatorics

László Lovász

Fall 2013

Contents

1 Preliminaries

1.1 Notation

The r-dimensional unit ball is denoted by B^r. Specifically, B^0 is a single point. Its boundary, the $(r-1)$-dimensional unit sphere is denoted by S^{r-1}. If two topological spaces T_1, T_2 are homeomorphic, then we write $T_1 \cong T_2$.

1.2 Simplicial complexes

A *simplicial complex* $\mathcal{K}$ is a finite collection of nonempty finite sets such that $X \in \mathcal{K}$, $\emptyset \neq Y \subseteq X$ implies $Y \in \mathcal{K}$. The union of all members of $\mathcal{K}$ is denoted by $V(\mathcal{K})$. The elements of $V(\mathcal{K})$ are called the *vertices* of $\mathcal{K}$, the elements of $\mathcal{K}$ are called the *simplices* of $\mathcal{K}$.

The *dimension* of a simplex $S \in \mathcal{K}$ is $\dim(S) = |S| - 1$. The *dimension* of $\mathcal{K}$ is the maximum dimension of any simplex in $\mathcal{K}$. The *k-dimensional skeleton* of a simplicial complex $\mathcal{K}$ is the simplicial complex $\mathcal{K}|_k = \{S \in \mathcal{K} : |X| \leq k+1\}$.

Let $n = |V(\mathcal{K})|$. Embed $V(\mathcal{K})$ in $\mathbb{R}^{n-1}$ so that the n points representing the vertices are not contained in one hyperplane. For each simplex S, take the convex hull $\mathrm{conv}(S)$ of the elements in S, and take the union of these simplices. This set $G(\mathcal{K}) \subset \mathbb{R}^{n-1}$ is called a *geometric realization* of $\mathcal{K}$. The sets $\mathrm{conv}(S)$ $(S \in \mathcal{K})$ are the *faces* of $G(\mathcal{K})$. By the "general position" assumption on the vertices, it follows that the faces intersect each other in their common subface only: $\mathrm{conv}(S_1)\cap\mathrm{conv}(S_2) = \mathrm{conv}(S_1\cap S_2)$. It is also clear that all geometric realizations of the same simplicial complex are homeomorphic.

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two simplicial complexes and let $f:\ V(\mathcal{K}_1) \to V(\mathcal{K}_2)$ be a mapping. We say that f is *simplicial* if $f(S) \in \mathcal{K}_2$ for every simplex $S \in \mathcal{K}_1$. Every simplicial mapping can be extended linearly to get a continuous mapping $\widehat{f}:\ G(\mathcal{K}_1) \to G(\mathcal{K}_2)$.

The *baricentric subdivision* $\mathcal{B}(\mathcal{K})$ of a simplicial complex $\mathcal{K}$ is obtained as follows. We create a new vertex v_S for every non-empty simplex S (the "baricenter" of S), and define $v_{S_1},\dots,v_{S_k}$ to be a simplex for every $S_1 \subset \cdots \subset S_k$. It is easy to see that $G(\mathcal{B}(\mathcal{K}))$ is homeomorphic with $G(\mathcal{K})$. In fact, there is a "canonical" homeomorphism $\beta:\ G(\mathcal{B}(\mathcal{K})) \to G(\mathcal{K})$, where $\beta(v_S)$ is the baricenter of $\mathrm{conv}(S)$ (which is a face of $G(\mathcal{K})$), and β is extended linearly over each face in $G(\mathcal{B}(\mathcal{K}))$.

1.3 Homotopy

Let T_1 and T_2 be two topological spaces. Two continuous maps $f_0, f_1 :\ T_1 \to T_2$ are called *homotopic* (denoted $f_0 \simeq f_1$), if they can be deformed into each other. More exactly, there exists a continuous mapping $F:\ T_1 \times [0,1] \to T_2$ such that $f_0(x) = F(x,0)$ and $f_1(x) = F(x,1)$.

We say that T_1 and T_2 are *homotopy equivalent* (denoted by $\mathcal{K}_1 \simeq \mathcal{K}_2$), if there exist continuous maps $f:\ T_1 \to T_2$ and $g:\ T_2 \to T_1$ such that $f\circ g \simeq \mathrm{id}_{T_2}$ and $g \circ f \simeq \mathrm{id}_{T_1}$.

Let $T_1 \subseteq T_2$ be two topological spaces. We say that T_1 is a *retract* of T_2 if there is a continuous map $\varphi:\ T_2 \to T_1$ such that $\varphi|_{T_1} = \mathrm{id}_{T_1}$. If in addition $\varphi \sim \mathrm{id}_{T_2}$ (as maps $T_2 \to T_2$), then we call T_1 a *deformation retract* of T_2.

It is easy to see that every deformation retract of a space is homotopy equivalent to it. This does not remain valid for retracts in general, but every retract of a contractible space is contractible.

A topological space T is *contractible* if it is homotopy equivalent to the space with a single point. This is equivalent to saying that the map $T \to \{p\}$ (where $p \in T$) is homotopic with the identity map id_T. This property does not depend on the choice of p.

A topological space is *k-connected* ($k \geq 0$) if for every $0 \leq r \leq k$, every continuous map $f:\ S^r \to T$ extends to a continuous map $\overline{f}:\ B^{r+1} \to T$. Equivalently, f is homotopic to a constant map. Sometimes it is convenient to define (-1)-connected as "non-empty".

We say that a simplicial complex is contractible [k-connected] if its geometric realization is. We define homotopy equivalence of two simplicial complexes and homotopy of two simplicial maps between simplicial complexes in a similar way.

Example 1.1 Any two continuous maps from a topological space into a contractible space are homotopic. Every convex body is contractible. A simple graph (as a 1-dimensional simplicial complex) is contractible if and only if it is a tree.

Any face of a convex polytope is a retract of the polytope. The 2-point space consisting of the endpoints of a segment is not a retract of this segment. An arc of a circle is a retract, but not a deformation retract, of the full circle.

If a simplicial complex has a vertex u that is contained in every maximal simplex, then it is contractible (just contract its geometric realization homothetically from center u).

Example 1.2 The *full simplex* on a finite set V is the simplicial complex $\Sigma(V) = 2^V \setminus \{\emptyset\}$. The *boundary* of the full simplex $\Sigma(V)$ is the simplicial complex $\partial(V) = 2^V \setminus \{\emptyset, V\}$. The usual geometric representation of $\Sigma(V)$ is the standard simplex $\Delta^V = \{x \in \mathbb{R}^V :\ x_i \geq 0,\ \sum_i x_i \leq 1\}$. Clearly, $G(\Sigma(V)) \cong B^{|V|}$ and $G(\partial(V)) \cong S^{|V|}$.

Example 1.3 The following example will be useful later on. Let $A_1,\dots,A_m$ be disjoint nonempty sets, and let $\mathcal{M}(A_1,\dots,A_m)$ consist of those subsets $X \subseteq A_1 \cup \dots A_m$ for which $|X \cap A_i| \leq 1$ for every $1 \leq i \leq m$. Then $\mathcal{M}(A_1,\dots,A_m)$ is a simplicial complex.

An interesting special case is obtained when $A_i = \{u_i, v_i\}$ has two elements. In this case the simplices of $\mathcal{M}(A_1, \dots, A_m)$ can be identified with the proper faces of the m-dimensional cross-polytope $X_n = \{x \in \mathbb{R}^n : \sum_i |x_i| \le 1\}$. So $\mathcal{M}(A_1, \dots, A_m) \cong S^{m-1}$.

Let $A'_i \subseteq A_i$, then $\mathcal{M}(A'_1, \dots, A'_m) \subseteq \mathcal{M}(A_1, \dots, A_m)$. In fact, $\mathcal{M}(A'_1, \dots, A'_m)$ is a retract of $\mathcal{M}(A_1, \dots, A_m)$: we can take any surjective map $\varphi_i : A_i \to A'_i$ for $i = 1, \dots, m$, then $\varphi = \varphi_1 \cup \dots \cup \varphi_m$ is a simplicial map $\mathcal{M}(A_1, \dots, A_m) \to \mathcal{M}(A'_1, \dots, A'_m)$ that is a retraction.

2 Combinatorial homotopy theory

2.1 Contractibility

The bad news: given a simplicial complex, it is algorithmically undecidable whether it is contractible. This implies that it is algorithmically undecidable whether two simplicial complexes are homotopy equivalent, or two simplicial maps are homotopic. It is somewhat surprising that in spite of these facts, there is a useful combinatorial theory of homotopy equivalence and contractibility.

Lemma 2.1 *Let T be a topological space and let $\mathcal{K}$ be a simplicial complex. Let $f, g : T \to G(\mathcal{K})$ be two continuous maps such that $f(x)$ and $g(x)$ are contained in the same face of $G(\mathcal{K})$ for all $x \in T$. Then $f \sim g$.*

Proof. $\Phi(t, x) = (1-t)f(x) + tg(x) \in G(\mathcal{K}_2)$ defines a homotopy of f and g. □

Lemma 2.2 *For a simplicial complex $\mathcal{K}$ with $V(\mathcal{K}) = V$, the following are equivalent:*

(i) $\mathcal{K}$ is contractible.

(ii) $\mathcal{K}$ is k-connected for every $k \ge 0$.

(iii) $\mathcal{K}$ is a retract of $\Sigma(V)$.

Proof. (i)⇒(ii): Trivial.

(ii)⇒(iii): We prove by induction on n that for every simplicial complex $\mathcal{K}' \supseteq \mathcal{K}$ with $V(\mathcal{K}') = V$ and $|\mathcal{K}'| = n$, $G(\mathcal{K}')$ has a retraction to $G(\mathcal{K})$. For $n = 2^{|V|}$ we get the assertion.

Let S be a maximal simplex in $\mathcal{K}'$ such that $S \notin \mathcal{K}$, and let $\mathcal{L} = \mathcal{K}' \setminus \{S\}$. By induction, we have a retract $\varphi : G(\mathcal{L}) \to G(\mathcal{K})$. Then φ gives a continuous map $\varphi_0 : \partial(G(\Sigma(S))) \to G(\mathcal{K})$. By k-connectivity, this extends to a continuous map $\varphi_1 : G(\Sigma(S)) \to G(\mathcal{K})$. Then $\varphi \cup \varphi_1 : G(\mathcal{K}') \to G(\mathcal{K})$ is a retraction.

(iii)⇒(i): $\Sigma(V)$ is contractible, and so every retract of it is contractible as well. □

Lemma 2.3 *Let $\mathcal{K}$ be a simplicial complex and $U \subseteq V(\mathcal{K})$. Suppose that $\mathcal{K} \cap \Sigma(U)$ is contractible. Then $\mathcal{K} \cup \Sigma(U) \sim \mathcal{K}$.*

Proof. By Lemma 2.2, there is a retraction $\varphi_0 : G(\Sigma(U)) \to G(\mathcal{K} \cap \Sigma(U))$. Define $\varphi : G(\mathcal{K} \cup \Sigma(U)) \to G(\mathcal{K} \cup \Sigma(U))$ by

$$\varphi(x) = \begin{cases} \varphi_0(x) & \text{if } x \in G(\Sigma(U)), \\ x, & \text{otherwise.} \end{cases}$$

Clearly $\varphi(x) \in G(\mathcal{K})$ for every x, and φ, as a map $G(\mathcal{K} \cup \Sigma(U)) \to G(\mathcal{K})$, is a retraction. Since x and $\varphi(x)$ are contained in the same simplex of $\mathcal{K} \cup \Sigma(U)$, it follows that $\varphi \sim \mathrm{id}_{G(\mathcal{K} \cup \Sigma(U))}$. Hence φ is a deformation retraction, showing that $\mathcal{K} \cup \Sigma(U) \sim \mathcal{K}$. □

This lemma generalizes to adding more than one simplex.

Lemma 2.4 *Let $\mathcal{K}$ be a simplicial complex and let $U_1, \dots, U_m \subseteq V(\mathcal{K})$. Suppose that for every $1 \leq i_1 < \cdots < i_r \leq m$, the restriction $\mathcal{K} \cap \Sigma(U_{i_1} \cap \cdots \cap U_{i_r})$ is either empty or contractible. Then $\mathcal{K} \sim \mathcal{K} \cup \Sigma(U_1) \cup \cdots \cup \Sigma(U_m)$.*

Proof. By induction on m. If $m = 0$, then we have nothing to prove. Let $m \geq 1$. By Lemma 2.3, we have $\mathcal{K} \sim \mathcal{K}' = \mathcal{K} \cup \Sigma(U_1)$. Furthermore,

$$\mathcal{K}' \cap \Sigma(U_{i_1} \cap \cdots \cap U_{i_r}) = (\mathcal{K} \cap \Sigma(U_{i_1} \cap \cdots \cap U_{i_r})) \cup \Sigma(U_1 \cap U_{i_1} \cap \cdots \cap U_{i_r})$$

is either empty or contractible for every $2 \leq i_1 < \cdots < i_r \leq m$, again by Lemma 2.3. By the induction hypothesis, this implies that

$$\mathcal{K}' \sim \mathcal{K}' \cup \Sigma(U_2) \cup \cdots \cup \Sigma(U_m) = \mathcal{K} \cup \Sigma(U_1) \cup \cdots \cup \Sigma(U_m).$$

This proves the Lemma. $\square$

Lemma 2.5 *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be contractible simplicial complexes. If $\mathcal{K}_1 \cap \mathcal{K}_2$ is contractible, then so is $\mathcal{K}_1 \cup \mathcal{K}_2$.*

Proof. Let $V_i = V(\mathcal{K}_i)$, $V = V_1 \cup V_2$ and $S = V_1 \cap V_2$. First, assume that $\mathcal{K}_1 \cap \mathcal{K}_2$ is a single simplex. The complex $\Sigma(V_1) \cup \Sigma(V_2)$ is contractible, hence by Lemma 2.2 there is a retraction $\varphi : G(\Sigma(V)) \to G(\Sigma(V_1)) \cup G(\Sigma(V_2))$. Similarly, $\mathcal{K}_i$ is contractible, and so there is a retraction $\varphi_i : G(\Sigma(V_i)) \to G(\mathcal{K}_i)$. Since $\Sigma(S) \subseteq \mathcal{K}_i$, we have $\varphi_i|_{G(\Sigma(S))} = \mathrm{id}_{G(\Sigma(S))}$, and so $\varphi_1 \cup \varphi_2$ is a well-defined continuous map $G(\Sigma(V_1)) \cup G(\Sigma(V_2)) \to G(\mathcal{K}_1) \cup G(\mathcal{K}_2) = G(\mathcal{K}_1 \cup \mathcal{K}_2)$, which is clearly a retraction. So $(\varphi_1 \cup \varphi_2) \circ \varphi : G(\Sigma(V)) \to G(\mathcal{K}_1 \cup \mathcal{K}_2)$ is a retraction, which proves that $\mathcal{K}_1 \cup \mathcal{K}_2$ is contractible by Lemma 2.2.

Next, we treat the more general (but still not completely general) case when $\mathcal{K}_1 \cup \Sigma(S) = \mathcal{K}_2 \cup \Sigma(S)$. Then $\mathcal{K}_1 \cup \Sigma(S)$, $\mathcal{K}_1 \cup \Sigma(S)$ and $(\mathcal{K}_1 \cap \mathcal{K}_2) \cup \Sigma(S)$ are contractible by Lemma 2.5, and hence by the special case proved above, so is $(\mathcal{K}_1 \cup \Sigma(S)) \cup (\mathcal{K}_2 \cup \Sigma(S)) = \mathcal{K}_1 \cup \mathcal{K}_2 \cup \Sigma(S)$. By Lemma 2.5, this implies that $\mathcal{K}_1 \cup \mathcal{K}_2$ is contractible.

Finally, in the general case, consider the baricentric subdivisions $\mathcal{B}(\mathcal{K}_i)$. These are contractible, and so is $\mathcal{B}(\mathcal{K}_1) \cap \mathcal{B}(\mathcal{K}_2) = \mathcal{B}(\mathcal{K}_1 \cap \mathcal{K}_2)$. Furthermore, we have $V(\mathcal{B}(\mathcal{K}_1)) \cap V(\mathcal{B}(\mathcal{K}_2)) = \mathcal{K}_1 \cap \mathcal{K}_2$, and the restriction of $\mathcal{B}(\mathcal{K}_i)$ to this is $\mathcal{B}(\mathcal{K}_1 \cap \mathcal{K}_2)$ for $i = 1, 2$. Hence the previous special case applies and we get that $\mathcal{B}(\mathcal{K}_1) \cup \mathcal{B}(\mathcal{K}_2) = \mathcal{B}(\mathcal{K}_1 \cup \mathcal{K}_2)$ is contractible. This implies that $\mathcal{K}_1 \cup \mathcal{K}_2$ is contractible. $\square$

Let $\mathcal{K}$ be a simplicial complex, and let $f : U \to V(\mathcal{K})$ be any mapping. Define the simplicial complex

$$f^{-1}(\mathcal{K}) = \{X \subseteq U : f(X) \in \mathcal{K}\}.$$

Lemma 2.6 *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be simplicial complexes and let $f : V(\mathcal{K}_1) \to V(\mathcal{K}_2)$ be a surjective simplicial map. Suppose that for every $S \in \mathcal{K}_2$, the complex $\mathcal{K}_1 \cap \Sigma(f^{-1}(S))$ is contractible. Then $\mathcal{K}_1 \sim \mathcal{K}_2$.*

Proof. We start with proving the special case when $\mathcal{K}_1 = f^{-1}(\mathcal{K}_2)$. In this case $f^{-1}(S) \in \mathcal{K}_1$ for every $S \in \mathcal{K}_2$, so the contractibility condition is trivially satisfied. By the definition of $f^{-1}(\mathcal{K}_2)$, the mapping $f : V(\mathcal{K}_1) \to V(\mathcal{K}_2)$ is simplicial, and so it extends to a continuous mapping $\widehat{f} : G(\mathcal{K}_1) \to G(\mathcal{K}_2)$.

For every $u \in V(\mathcal{K}_2)$, let $g(u) \in G(\mathcal{K}_1)$ be the center of gravity of the simplex $f^{-1}(u) \in \mathcal{K}_1$. If $S = \{u_1, \dots, u_d\} \in \mathcal{K}_2$, then $f^{-1}(S) \in \mathcal{K}_1$, and hence $g(u_1), \dots, g(u_d)$ are contained in the face $\mathrm{conv}(f^{-1}(S))$ of $G(\mathcal{K}_1)$. Hence we can extend the map g linearly to a continuous map $\widehat{g} : G(\mathcal{K}_2) \to G(\mathcal{K}_1)$.

It is clear that $\widehat{g} \circ \widehat{f} = \mathrm{id}_{G(\mathcal{K}_2)}$. On the other hand, let $x \in G(\mathcal{K}_1)$, say $x \in \mathrm{conv}(S)$, where $S \in \mathcal{K}_1$. Then both x and $\widehat{f}(\widehat{g}(x))$ are contained in the face $\mathrm{conv}(f^{-1}(f(S))$, and hence $\widehat{f} \circ \widehat{g} \sim \mathrm{id}_{G(\mathcal{K}_1)}$ by Lemma 2.1. This proves that $\mathcal{K}_1 \sim \mathcal{K}_2$.

Now in the general case, we already know that $f^{-1}(\mathcal{K}_2) \sim \mathcal{K}_2$. Let $S_1, \dots, S_m$ be the maximal simplices in $\mathcal{K}_2$, and let $U_i = f^{-1}(S_i)$. Then

$$\mathcal{K}_1 \cap \Sigma(U_{i_1} \cap \dots \cap U_{i_r}) = \mathcal{K}_1 \cap \Sigma(f^{-1}(S_{i_1} \cap \dots \cap S_{i_r}))$$

is contractible for every $1 \le i_1 < \dots < i_r$, and hence by Lemma 2.4,

$$\mathcal{K}_1 \sim \mathcal{K}_1 \cup \Sigma(U_1) \cup \dots \cup \Sigma(U_m) = \Sigma(U_1) \cup \dots \cup \Sigma(U_m) = f^{-1}(\mathcal{K}_2)$$

Thus $\mathcal{K}_1 \sim f^{-1}(\mathcal{K}_2) \sim \mathcal{K}_2$. □

2.2 The Nerve Theorem

Let $G = (V, E)$ be a bipartite graph with bipartition $V = U \cup W$. The *neighborhood complex on* U $\mathcal{N}_U = \mathcal{N}_U(G)$ of G is the simplicial complex consisting of all subsets A of U such that the elements of A have a common neighbor. $\mathcal{N}_W = \mathcal{N}_W(G)$ is defined analogously.

Lemma 2.7 *The two neighborhood complexes of a bipartite graph are homotopy equivalent.*

Proof. For every simplex $S \in \mathcal{N}_U$, let $f(S) \subseteq W$ denote the set common neighbors of nodes in S. Clearly $f(S) \in \mathcal{N}_W$, so $f: \mathcal{N}_U \to \mathcal{N}_W$. We define $g: \mathcal{N}_W \to \mathcal{N}_U$ analogously.

If $S_1 \subset \dots \subset S_k$, then $f(S_1) \supset \dots \supset f(S_k)$, and hence f is simplicial as a map $V(\mathcal{B}(\mathcal{N}_U)) = \mathcal{N}_U \to V(\mathcal{B}(\mathcal{N}_W)) = \mathcal{N}_W$. Thus f defines a continuous map $\overline{f}: G(\mathcal{B}(\mathcal{N}_U)) \to G(\mathcal{B}(\mathcal{N}_W))$. We get a continuous map $\overline{g}: G(\mathcal{B}(\mathcal{N}_W)) \to G(\mathcal{B}(\mathcal{N}_U))$ analogously.

Let $\alpha: G(\mathcal{B}(\mathcal{N}_U)) \to G(\mathcal{N}_U)$ and $\beta: G(\mathcal{B}(\mathcal{N}_W)) \to G(\mathcal{N}_W)$ be the canonical homeomorphisms. We claim that the maps $\widehat{f} = \beta \circ \overline{f} \circ \alpha^{-1}$ and $\widehat{g} = \alpha \circ \overline{g} \circ \beta^{-1}$ establish the homotopy equivalence of $G(\mathcal{N}_U)$ and $G(\mathcal{N}_W)$.

Consider the map $h = \widehat{g} \circ \widehat{f} = \alpha \circ \overline{g} \circ \overline{f} \circ \alpha^{-1}: G(\mathcal{N}_U) \to G(\mathcal{N}_U)$. We claim that x and $h(x)$ belong to the same simplex for every $x \in G(\mathcal{K}_1)$. Indeed, let $S \in \mathcal{N}_U$ be a smallest simplex such that $x \in \mathrm{conv}(S)$. Then $\alpha^{-1}(x) \in \mathrm{conv}\{S_1, \dots, S_k\}$ for a simplex in $\mathcal{B}(\mathcal{N}_U)$ such that $S_1 \subset \dots \subset S_k = S$. It follows that $\mathrm{conv}(g(f(S_1)), \dots, g(f(S_1)))$ contains $\overline{g}(\overline{f}(\alpha^{-1}(x)))$, and so $\alpha(\mathrm{conv}(g(f(S_1)), \dots, g(f(S_k))))$ contains $h(x)$. This (geometric) simplex is contained in $\mathrm{conv}(g(f(S)))$. Also, trivially $g(f(S)) \supseteq S$, and hence $x \in \mathrm{conv}(S) \subseteq \mathrm{conv}(f(g(S)))$. So Lemma 2.1 implies that h and $\mathrm{id}_{G(\mathcal{N}_U)}$ are homotopic.

We get similarly that $\widehat{f} \circ \widehat{g} \sim \mathrm{id}_{G(\mathcal{N}_W)}$, and so $G(\mathcal{N}_U) \sim G(\mathcal{N}_W)$ as claimed. □

Let $\mathcal{H}$ be a hypergraph consisting of nonempty sets. The *nerve* of $\mathcal{H}$ is defined as the simplicial complex $\mathsf{nerve}(\mathcal{H})$ whose vertices are the sets in $\mathcal{H}$, and $\{X_1, \dots, X_r\} \in \mathsf{nerve}(\mathcal{H})$ if and only if $X_1, \dots, X_r \in \mathcal{H}$ and $X_1 \cap \dots \cap X_r \ne \emptyset$.

Corollary 2.8 *Let $\mathcal{K}$ be a simplicial complex, and assume that $\mathcal{H} \subseteq \mathcal{K}$ contains all maximal simplices in $\mathcal{K}$. Then $\mathcal{K} \sim \mathsf{nerve}(\mathcal{H}) \sim \mathsf{nerve}(\mathcal{K})$.*

More generally:

Theorem 2.9 (Nerve Theorem) *Let $\mathcal{K}$ be a simplicial complex and let $\mathcal{K}_1, \dots, \mathcal{K}_m$ be subcomplexes such that $\mathcal{K} = \mathcal{K}_1 \cup \dots \cup \mathcal{K}_m$. Assume that $\mathcal{K}_{i_1} \cap \dots \cap \mathcal{K}_{i_r}$ is either empty or contractible for every $1 \le i_1 < \dots < i_r \le m$, $r \ge 1$. Then $\mathcal{K} \sim \mathsf{nerve}\{V(\mathcal{K}_1), \dots, V(\mathcal{K}_m)\}$.*

Proof. Let $V_i = V(\mathcal{K}_i)$. First, we prove the case when $\mathcal{K}_i = \mathcal{K} \cap \Sigma(V_i)$. In this case $\mathcal{K}_{i_1} \cap \dots \cap \mathcal{K}_{i_r} = \mathcal{K} \cap \Sigma(V_{i_1} \cap \dots \cap V_{i_r})$. Lemma 2.4 implies that $\mathcal{K} \sim \mathcal{K} \cup \Sigma(V_1) \cup \dots \cup \Sigma(V_m) =$

$\Sigma(V_1) \cup \cdots \cup \Sigma(V_m)$. By Corollary 2.8, we have $\Sigma(V_1) \cup \cdots \cup \Sigma(V_m) \sim \mathsf{nerve}\{V_1, \ldots, V_m\}$, which proves the theorem in this case.

For the general case (when $\mathcal{K}_i$ may be smaller than $\mathcal{K} \cap \Sigma(V_i)$), we replace each $\mathcal{K}_i$ by $\mathcal{B}(\mathcal{K}_i)$ as in the proof of Lemma 2.5. □

2.3 Topological connectivity

The lemmas from the previous sections can be generalized, with some care, from contractibility to k-connectivity.

Lemma 2.10 *Let $\mathcal{K}$ be a simplicial complex and $k \geq 0$. Then the following are equivalent:*

(a) *$\mathcal{K}$ is k-connected.*

(b) *The $(k+1)$-dimensional skeleton $\mathcal{K}|_{k+1}$ is k-connected.*

(c) *$\mathcal{K}|_{k+1}$ is a retract of $\Sigma(V(\mathcal{K}))|_{k+1}$.*

Lemma 2.11 *Let $\mathcal{K}$ be a simplicial complex and $U \subseteq V(\mathcal{K})$.*

(a) If $\mathcal{K}$ is k-connected and $\mathcal{K} \cap \Sigma(U)$ is $(k-1)$-connected, then $\mathcal{K} \cup \Sigma(U)$ is k-connected.

(b) If $\mathcal{K} \cup \Sigma(U)$ and $\mathcal{K} \cap \Sigma(U)$ are k-connected, then $\mathcal{K}$ is k-connected.

Lemma 2.12 *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two k-connected simplicial complexes and assume that $\mathcal{K}_1 \cap \mathcal{K}_2$ is $(k-1)$-connected. Then $\mathcal{K}_1 \cup \mathcal{K}_2$ is k-connected.*

The Nerve Theorem 2.9 has the following version for k-connectivity.

Theorem 2.13 (Connectivity Nerve Theorem) *Let $\mathcal{K}$ be a simplicial complex and let $\mathcal{K}_1, \ldots, \mathcal{K}_m$ be subcomplexes such that $\mathcal{K} = \mathcal{K}_1 \cup \cdots \cup \mathcal{K}_m$. Assume that $\mathcal{K}_{i_1} \cap \cdots \cap \mathcal{K}_{i_r}$ is either empty or $(k-r+1)$-connected for every $1 \leq i_1 < \cdots < i_r \leq m$, $1 \leq r \leq k+1$. Then $\mathcal{K}$ is k-connected if and only if the nerve of $\{V(\mathcal{K}_1), \ldots, V(\mathcal{K}_m)\}$ is k-connected.*

Example 2.14 As an application of this theorem, we analyze the connectivity of the simplicial complex $\mathcal{M}(A_1, \ldots, A_m)$ from Example 1.3. If $|A_i| = 1$ for some i, then every maximal simplex contains the single element of A_i, and hence $\mathcal{M}(A_1, \ldots, A_m)$ is contractible. If $|A_i| \geq 2$ for every i, then $\mathcal{M}(A_1, \ldots, A_m)$ is $(m-2)$-connected. This is trivial if $m \leq 1$, so we may assume that $m \geq 2$. Let $\mathcal{M}_x$ be the set of all simplices in $\mathcal{M}(A_1, \ldots, A_m)$ that contain x or can be extended to contain x. Then $\mathcal{M}_x$ is contractible by the argument above. Furthermore,

$$\mathcal{M}(A_1, \ldots, A_m) = \bigcup_{x \in A_1} \mathcal{M}_x.$$

Clearly

$$\mathcal{M}_{x_1} \cap \cdots \cap \mathcal{M}_{x_r} = \mathcal{M}(A_2, \ldots, A_m)$$

if $x_1, \ldots, x_r$ are distinct elements of A_1 ($r \geq 2$). The complex $\mathcal{M}_{x_1} \cap \cdots \cap \mathcal{M}_{x_r}$ is $(m-3)$-connected by induction. The Connectivity Nerve Theorem 2.13 applies, and gives that $\mathcal{M}(A_1, \ldots, A_m)$ is $(m-2)$-connected.

Example 2.15 Let $V = \{1, \ldots, n\}$, let $0 \leq a \leq b \leq n$, and consider the bipartite graph G between color classes $U = \binom{V}{a}$ and $W = \binom{V}{b}$, where $X \in U$ is adjacent to $Y \in W$ if and only if $X \subseteq Y$. We claim that the neighborhood complexes $\mathcal{N}_U = \mathcal{N}_U(G)$ and $\mathcal{N}_W = \mathcal{N}_W(G)$ are $(b-a-1)$-connected.

We know that $\mathcal{N}_U \sim \mathcal{N}_W$ (Lemma 2.7), so it suffices to prove that $\mathcal{N}_U$ is $(b-a-1)$-connected. This is trivial in some cases: if $a = b$ (recall that (-1)-connected means non-empty); if $a = 0$ (a one-point space is contractible); and if $b = n$ (a full simplex is contractible). So we may assume that $0 < a < b < n$.

For every $i \in V$, let $U_i = \binom{V \setminus \{i\}}{a}$ and let $\mathcal{K}_i = \mathcal{N}_U \cap \Sigma(U_i)$. Since $b < n$, we have $\mathcal{N}_U = \cup_i \mathcal{K}_i$. By induction on n, we know that $\mathcal{K}_i$ is $(b-a-1)$-connected.

More generally, let $1 \le i_1 < \cdots < i_r \le n$. If $r \le n-b$, then $\mathcal{K}_{i_1} \cap \cdots \cap \mathcal{K}_{i_r}$ is just the neighborhood complex of the bipartite graph between levels $\binom{V'}{a}$ and $\binom{V'}{b}$, where $V' = V \setminus \{i_1, \dots, i_r\}$. Hence by induction, $\mathcal{K}_{i_1} \cap \cdots \cap \mathcal{K}_{i_r}$ is $(b-a-1)$-connected. If $n-b \le r \le n-a$, then $\mathcal{K}_{i_1} \cap \cdots \cap \mathcal{K}_{i_r}$ is a full simplex, so it is contractible. If $r > n-a$, then $\mathcal{K}_{i_1} \cap \cdots \cap \mathcal{K}_{i_r} = \emptyset$. So the Connectivity Nerve Theorem 2.13 applies, and we get that $\mathcal{N}_U$ is $(b-a)$-connected.

An important use of connectivity is the following extension property of maps.

Lemma 2.16 *Let $\mathcal{K}$ is a $(k+1)$-dimensional simplicial complex, let $\mathcal{K}'$ be a subcomplex of $\mathcal{K}$, and T a k-connected space. Then every continuous map $f : G(\mathcal{K}') \to T$ extends to a continuous map $G(\mathcal{K}) \to T$.*

Proof. Let S be a simplex in $\mathcal{K} \setminus \mathcal{K}'$ with minimal dimension. Then $\partial(S) \subseteq \mathcal{K}'$ and so the map f is already defined on $G(\partial(S))$. Hence by the k-connectivity of T, f extends to a continuous map $G(\mathcal{K}' \cup \{S\}) \to T$. Repeating this argument, we extend f to a continuous map $G(\mathcal{K}) \to T$. □

3 Brouwer's fixed point theorem

A simplicial complex $\mathcal{K}$ such that $G(\mathcal{K}) \cong S^r$ is called a *triangulation* of the sphere S^r.

Lemma 3.1 (Sperner's lemma) *Let $\mathcal{K}$ be a triangulation of the sphere S^r. Let $f : V(\mathcal{K}) \to \{0, 1, \dots, r\}$ be any labeling of the vertices of S^r. Then the number of simplices whose vertices are labeled by different numbers is even.*

Theorem 3.2 (Brouwer's fixed point theorem) *Every continuous map $B^r \to B^r$ has a fixed point.* □

There are a number of equivalent facts.

Corollary 3.3 (a) *Let $\mathcal{K}$ be a contractible simplicial complex. Then every continuous map $G(\mathcal{K}) \to G(\mathcal{K})$ has a fixed point.*

(b) *S^{r-1} is not a retract of B^r.*

(c) *Let $f : \Sigma^r \to \Sigma^r$ be a continuous map such that for every face F of Σ^r, $f(F) \subseteq F$. Then f is surjective.*

We need the following group-theoretic corollary of Brouwer's Fixed Point Theorem.

Corollary 3.4 *Let T be a compact contractible topological space and let G be a finite cyclic group acting on T. Then the elements of G have a common fixed point.*

Proof. Let φ be the action of a generator of G. Then φ has a fixed point by Corollary 3.3, and this is a fixed point of every other element of G as well. □

This last corollary does not remain valid for all finite groups, but it does remain valid for certain non-cyclic finite groups. One such class of finite groups is the following.

Theorem 3.5 [41] *Let T be a compact contractible topological space and let Γ be a finite group acting on T. Assume that Γ has a normal p-subgroup Γ_1 such that Γ/Γ_1 is cyclic. Then the elements of G have a common fixed point.* $\square$

Example 3.6 To illustrate how to apply these results, let us return to Example 1.3. We show that if $|A_i| \geq 2$ for all $1 \leq i \leq m$, then $\mathcal{M}(A_1, \ldots, A_m)$ is not $(m-1)$-connected. Let $A'_i = \{u_i, v_i\} \subseteq A_i$. The embedding $\mathcal{M}(A'_1, \ldots, A'_m) \to \mathcal{M}(A_1, \ldots, A_m)$ gives continuous map of $f: \partial X_n \to G(\mathcal{M}(A_1, \ldots, A_m))$. Suppose that f extends to a map $\widehat{f}: X_n \to G(\mathcal{M}(A_1, \ldots, A_m))$. Since $\mathcal{M}(A'_1, \ldots, A'_m)$ is a retract of $\mathcal{M}(A_1, \ldots, A_m)$, we may assume that $\widehat{f}$ maps into $G(\mathcal{M}(A'_1, \ldots, A'_m)) = \partial X_n$. But this means that ∂X_n is a retract of X_n, a contradiction.

3.1 Evasive graph properties

Consider any property $\mathcal{P}_n$ of graphs with n nodes (we assume that if a graph has this property then every graph isomorphic with it also has it). We say that $\mathcal{P}_n$ *trivial*, if either every graph has it or no one has it. A graph property is *monotone* if whenever a graph has it each of its subgraphs has it. For most graph properties that we investigate (connectivity, the existence of a Hamiltonian circuit, the existence of perfect matching, colorability, etc.) either the property itself or its negation is monotone.

We say that a property $\mathcal{P}_n$ of n-node graphs is *evasive*, if it cannot be computed by a decision tree of depth less than $\binom{n}{2}$. The Aanderaa–Rosenberg–Karp conjecture says that *every nontrivial monotone graph property is evasive.* This conjecture is unsolved, but it has been proved for a number of graph properties. Here we prove the following

Theorem 3.7 (Kahn, Saks, Sturtevant) *Every nontrivial monotone property of graphs on n nodes, where n is a prime power, is evasive.*

The analogous theorem is also proved for bipartite graphs (Yao). See also `http://www.cs.elte.hu/~lovasz/complexity.pdf` and `http://arxiv.org/abs/cs/0205031` for other special cases, provable by topology and also by different tools.

Before proving this theorem, it will be useful to generalize it. We call a Boolean function $f: \{0,1\}^n \to \{0,1\}$ *evasive*, if it cannot be computed by a decision tree of depth less than n. We define the *automorphism group* of a Boolean function as the set of permutations of the variables that does not change the value of the function. Recall that this permutation group is called *transitive*, if for any two variables x and y there is an automorphism that moves x to y. We say that the function f is *monotone decreasing* (or just "monotone" for this section), if $f(x_1, \ldots, _n) \leq f(y_1, \ldots, y_n)$ whenever $x_i \geq y_i$ for all $1 \leq i \leq n$.

To see how graph properties fit in, let G be graph with $V = V(G) = \{1, \ldots, n\}$. For every pair $i, j \subseteq V$, let us introduce a Boolean variable x_{ij} with value 1 if i and j are adjacent and 0 if they are not. In this way, any property of n-point graphs can be considered as a Boolean function with $\binom{n}{2}$ variables. This Boolean function has a transitive automorphism group: for any two pairs of nodes there is a permutation of the nodes taking one into the other, which leads to an isomorphic graph and hence does not change the value of the Boolean function.

The following Generalized Aanderaa–Rosenberg–Karp Conjecture is open: *If the automorphism group of a non-constant monotone Boolean function is transitive, then the function evasive.* We prove it in a special case.

Theorem 3.8 *If the automorphism group of a non-constant monotone Boolean function f contains a transitive subgroup Γ, and Γ has a normal subgroup Δ of prime power order such that Γ/Δ is cyclic, then f is evasive.*

Corollary 3.9 *If the automorphism group of a non-constant monotone Boolean function is transitive, and it is a cyclic group, then it is evasive.*

Corollary 3.10 *If the automorphism group of a non-constant monotone Boolean function is transitive, and the number of its variables is a prime power, then it is evasive.*

Next, we describe how this problem relates to topology. Let $f:\ \{0,1\}^n \to \{0,1\}$ be a monotone decreasing Boolean function that is not identically 0. For $x \in \{0,1\}^n$, let $\mathrm{supp}(x) = \{i \in \{1,\dots,n\}:\ x_i = 1\}$. We associate with f the following simplicial complex:

$$\mathcal{K}_f = \bigl\{\mathrm{supp}(x):\ x \in \{0,1\}^n,\ x \neq 0,\ f(x) = 1\bigr\}.$$

Lemma 3.11 *If f is non-evasive, then $\mathcal{K}_f$ is contractible.*

Proof. If f is non-evasive, then there is decision tree computing f with depth at most $n-1$. This decision tree starts with checking a variable, say x_1. The two branches of the tree compute the Boolean functions $f_0(x_2,\dots,x_n) = f(0,x_2,\dots,x_n)$ and $f_1(x_2,\dots,x_n) = f(1,x_2,\dots,x_n)$. Since the depth of these branches is at most $n-2$, the functions f_0 and f_1 are non-evasive. By induction, we may assume that $\mathcal{K}_{f_0}$ and $\mathcal{K}_{f_1}$ are contractible.

Let $\mathcal{K}'_{f_1} = \mathcal{K}_{f_1} \cup \{X \cup \{1\}:\ X \in \mathcal{K}_{f_1}\}$, then all maximal simplices of $\mathcal{K}'_{f_1}$ contain 1, and so $\mathcal{K}'_{f_1}$ is contractible. We have $\mathcal{K}_{f_0} \cup \mathcal{K}'_{f_1} = \mathcal{K}$ and $\mathcal{K}_{f_0} \cap \mathcal{K}'_{f_1} = \mathcal{K}_{f_1}$, and so Lemma 2.5 implies that $\mathcal{K}$ is contractible. ☐

Next, we observe the following.

Lemma 3.12 *If f is a nonconstant monotone Boolean function with a transitive automorphism group Γ, then Γ acts on $G(\mathcal{K}_f)$ and has no fixed point.*

Proof. It is clear that every automorphism γ of f is a simplicial map $\gamma:\ \mathcal{K}_f \to \mathcal{K}_f$, and this extends to a continuous map $\widehat{\gamma}:\ G(\mathcal{K}_f) \to G(\mathcal{K}_f)$.

We claim that these maps have no common fixed point. Suppose that $x \in G(\mathcal{K}_f)$ satisfies $\widehat{\gamma}(x) = x$ for every $\gamma \in \Gamma$. There is a unique smallest simplex S such that $x \in \mathrm{conv}(S)$. Then $x = \widehat{\gamma}(x) = \in \mathrm{conv}(\gamma(S))$, so by the minimality of S, we must have $\gamma(S) = S$. Since the group γ is transitive, this can only happen for every $\gamma \in \Gamma$ if $S = V(\mathcal{K}_f)$. But then the function is identically true. ☐

Proof of Theorem 3.8. If f is non-evasive, then $G(\mathcal{K}_f)$ is contractible by Lemma 3.11. If f is nontrivial, then Γ acts on $G(\mathcal{K}_f)$ without fixed points. But this contradicts the following theorem of Oliver from topology: *If Γ is a finite group that has a normal subgroup Δ of prime power order such that Γ/Δ is cyclic, then every continuous action of Γ on a contractible simplicial complex has a fixed point.* ☐

Proof of Theorem 3.7. We consider $V(G)$ as the set of elements of a finite field $\mathbb{F}_q$ (where q is any prime poser), and define Γ as the group of linear transformations of the form $x \mapsto ax+b$, where $a, b \in Fbb_q$ and $a \neq 0$. Then Γ acts on pairs $\{i,j\} \subseteq \mathbb{F}_q$ transitively. Furthermore, if Δ denotes the subgroup of Γ of transformations of the form $x \mapsto x + b$, then $|\Delta| = q$ is a prime power, and $\Gamma/\Delta \cong F_q^*$ (the multiplicative group of nonzero elements of $\mathbb{F}_q$), which is cyclic by basic results in algebra. So Theorem 3.8 applies and proves the theorem. ☐

4 The Borsuk–Ulam Theorem

Let T be a topological space and let ι be an antipodality on T, i.e., a homeomorphism $\iota:\ T \to T$ such that ι is an involution ($\iota \circ \iota = \mathrm{id}_T$) and ι has no fixed points. Sometimes we denote $\iota(x)$ by $-x$. T, together with the antipodality ι, is called an *antipodality space.*

An *antipodal map* between antipodality spaces T_1 and T_2 is a continuous map $f:\ T_1$ to T_2 that satisfies $f(-x) = -f(x)$.

Lemma 4.1 *If T is a k-connected topological space with an antipodality, then there exists an antipodal map of S^{k+1} into T.*

The three assertions in the following theorem are all essentially equivalent to each other, and are called the *Borsuk–Ulam Theorem.*

Theorem 4.2
(a) *There is no antipodal map from S^r into S^{r-1} ($r \geq 1$).*
(b) *For every continuous map $f : S^r \to \mathbb{R}^r$ there exists an $x \in S^r$ such that $f(x) = f(-x)$.*
(c) *If S^r is covered by $r+1$ sets, and either each of these is closed or each of these is open, then one of these sets contains an antipodal pair.*

The Borsuk–Ulam theorem has the following following discrete version. Let P be a full-dimensional convex polytope in $\mathbb{R}^d$. We say that two faces A and B of P are *opposite* if there exists a linear function $\ell : \mathbb{R}^d \to \mathbb{R}$ that is maximized by the set of points in A and minimized by the set of points in B.

Theorem 4.3 [6] *Let P be a full-dimensional convex polytope in $\mathbb{R}^d$ and $f : P \to \mathbb{R}^{d-1}$. Then P has two opposite faces A and B such that $f(A) \cap f(B) \neq \emptyset$.*

The following slight extension of this theorem will be needed later. We say that the map $f : P \to \mathbb{R}^{d-1}$ is *generic* if for every pair of faces A and B with $\dim(A) + \dim(B) < d-1$, we have $f(A) \cap f(B) = \emptyset$, and for every pair of faces A and B with $\dim(A) + \dim(B) = d-1$, the set $f(A) \cap f(B)$ is finite.

Theorem 4.4 [36] *Let P be a full-dimensional convex polytope in $\mathbb{R}^d$ and let $f : P \to \mathbb{R}^{d-1}$ be a generic map. Then P has two opposite faces A and B such that* $\dim(A) + \dim(B) = d-1$ *and $|f(A) \cap f(B)|$ is odd.*

We say that the polytope P in $\mathbb{R}^d$ is *generic*, if for every two opposite faces A and B, $\dim(A) + \dim(B) = d-1$.

Theorem 4.5 *Let P be a generic full-dimensional convex polytope in $\mathbb{R}^d$ and let $f : P \to \mathbb{R}^{d-1}$ be a generic map. Then*

$$\sum |f(A) \cap f(B)| \equiv 1 \qquad (\mathrm{mod} 2),$$

where the summation extends over all pairs of opposite faces A and B of P.

4.1 Simplicial complexes associated with graphs

Let $G = (V, E)$ be any graph. The *neighborhood complex* $\mathcal{N}(G)$ of G is the simplicial complex consisting of all subsets A of V such that the elements of A have a common neighbor.

For a bipartite graph, the neighborhood complex consists of two components.

The set $\mathrm{Hom}(F, G)$ of homomorphisms $F \to G$ can also be equipped with a topological structure. We say that a set of homomorphisms $\varphi_1, \dots, \varphi_k : F \to G$ is a *cluster* if for every edge $uv \in E(F)$ and any $1 \leq i < j \leq k$, we have $\varphi_i(u)\varphi_j(v) \in E(G)$. It is clear that these clusters form a simplicial complex $\mathbf{Hom}(F, G)$ (i.e., they are closed under taking subsets).

This construction is "functorial", which means that every homomorphism $\psi : G_1 \to G_2$ induces a simplicial (and hence continuous) map $\widehat{\psi} : \mathbf{Hom}(F, G_1) \to \mathbf{Hom}(F, G_2)$ in a canonical way: For every homomorphism $\varphi : F \to G_1$, we define $\widehat{\psi}(\varphi) = \varphi\psi$. It is trivial that this map from $V\big(\mathbf{Hom}(F, G_1)\big) = \mathrm{Hom}(F, G_1)$ to $V\big(\mathbf{Hom}(F, G_2)\big) = \mathrm{Hom}(F, G_2)$ maps clusters onto clusters. Similarly, $\mathbf{Hom}(.,.)$ is a "contravariant functor" in its first variable, which means that every homomorphism $\xi : F_1 \to F_2$ induces a simplicial map $\check{\varphi} : \mathbf{Hom}(F_2, G) \to \mathbf{Hom}(F_1, G)$.

The complex $\mathbf{Hom}(K_2, G)$ plays a special role. We will denote it by $\mathcal{H}(G)$. The points of $\mathcal{H}(G)$ can be thought of as ordered pairs (u, v), where u and v are adjacent nodes of G. This space has a natural antipodality: $\iota(u, v) = (v, u)$ defines a simplicial map that extends to a bijective and fixed-point-free involution $\widehat{\iota}: G(\mathcal{H}(G)) \to G(\mathcal{H}(G))$. If $G_1 \to G_2$ is a graph homomorphism, then the map $\mathbf{Hom}(K_2, G_1) \to \mathbf{Hom}(K_2, G_2)$ it induces is antipodal.

Example 4.6 The complex $\mathcal{N}(K_n)$ is homeomorphic to the $(n-2)$-dimensional sphere: $\mathcal{N}(K_n) = \Sigma(V) \setminus \{V\}$ (here $V = V(K_n) = \{1, \dots, n\}$).

The complex $\mathcal{H}(K_n)$ is homotopy equivalent to the $(n-2)$-dimensional sphere S^{n-2}. Define a map $f: G(\mathcal{H}(K_n)) \to \mathbb{R}^n$ as follows. Let $e_1, \dots, e_n$ be the standard basis in $\mathbb{R}^n$. For $(u, v) \in V(\mathcal{H}(K_n))$, let $f(u, v) = e_i - e_j$. Extend f linearly over $G(\mathcal{H}(K_n))$.

We claim that the origin is not in the range of f. Indeed, consider a point $x \in \mathrm{conv}((u_1, v_1), \dots, (u_m, v_m))$, where $\{(u_1, v_1), \dots, (u_m, v_m)\} \in \mathcal{H}(K_n)$. Note that this implies that $u_i \neq v_j$ for $1 \le i, j \le m$. If $f(x) = 0$, then

$$\sum_{i=1}^{m} a_i (e_{u_i} - e_{v_j}) = 0$$

for some $a_i \ge 0$, $\sum_i a_i = 1$. We may assume that $a_1 > 0$, then the inner product of the left side with e_{u_1} is positive, which is a contradiction.

It follows that $x \mapsto f(x)/\|f(x)\|$ is well defined, and gives an antipodal map $G(\mathcal{H}(K_n)) \to S^{n-1}$. Since the range is contained in the hyperplane $H = \{x : \sum_i x_i = 0\}$, this map goes into $S^{n-1} \cap H$, which is a copy of S^{n-2}.

The construction of the inverse map is not given here (we will need this map only).

It is not a coincidence that $\mathcal{H}(K_n)$ and $\mathcal{N}(K_n)$ turned out to be homotopically equivalent, as this is shown by the following lemma.

Lemma 4.7 *For every graph G, the complexes $\mathcal{H}(G)$ and $\mathcal{N}(G)$ are homotopy equivalent.*

Proof. We define a simplicial map $f: V(\mathcal{H}(G)) \to V(\mathcal{N}(G))$ as follows: For every $(u, v) \in V(\mathcal{H}(G))$ we let $f(u, v) = u$. It suffices to show that this map satisfies the conditions of Lemma 2.6: for every simplex $S \in \mathcal{N}(G)$, the complex $\mathcal{H}(G) \cap \Sigma(f^{-1}(S))$ is contractible.

For every $T \in \mathcal{N}(G)$, let $Q(T)$ denote the set of nodes adjacent to every node of T. Then $Q(T) \neq \emptyset$ by definition, and $T \times Q(T) \in \mathcal{H}(G)$. For a given $S \in \mathcal{N}(G)$, we have

$$\mathcal{H}(G) \cap \Sigma(f^{-1}(S)) = \bigcup_{T \in \Sigma(S)} \Sigma(T \times Q(T)), \tag{1}$$

and hence by Corollary 2.8,

$$\mathcal{H}(G) \cap \Sigma(S \times Q(S)) \sim \mathsf{nerve}\{T \times Q(T) : T \in \Sigma(S)\} \tag{2}$$

Now it is easy to see that for $T_1, \dots, T_r \subseteq S$, we have

$$T_1 \times Q(T_1) \cap \dots \cap T_r \times Q(T_r) \neq \emptyset \quad \Leftrightarrow \quad T_1 \cap \dots \cap T_r \neq \emptyset. \tag{3}$$

Indeed, the implication $\Rightarrow$ is trivial, and if $T_1 \cap \dots \cap T_r \neq \emptyset$ than $(u, v) \in T_1 \times Q(T_1) \cap \dots \cap T_r \times Q(T_r)$ for any $u \in T_1 \cap \dots \cap T_r$ and $v \in Q(S)$. Thus

$$\mathsf{nerve}\{T \times Q(T) : T \in \Sigma(S)\} = \mathsf{nerve}(\Sigma(S)) \sim \Sigma(S).$$

Since $\Sigma(S)$ is contractible, this proves the Lemma. $\square$

Remark 4.8 It would be tempting to try a shorter argument and use the formula

$$\mathcal{H}(G) = \bigcup_{T\in\mathcal{N}(G)} \Sigma(T \times Q(T)),$$

which implies

$$\mathcal{H}(G) \sim \mathsf{nerve}\{T \times Q(T) :\ T \in \mathcal{N}(G)\}.$$

But the nerve on the right is difficult to handle, since (3) does not remain valid if $T_1, \ldots, T_r$ are not faces of the same simplex.

4.2 Topological connectivity and chromatic number

Theorem 4.9 [32] *If the neighborhood complex $\mathcal{N}(G)$ of a graph G is k-connected (equivalently, the homomorphism complex $\mathbf{Hom}(K_2, G)$ is k-connected), then $\chi(G) \geq k + 3$.*

Proof. By Lemma 4.1 there is an antipodal map $S^{k+1} \to \mathbf{Hom}(K_2, G)$. Suppose that $\chi(G) \leq k + 2$, then there is a homomorphism $G \to K_{k+2}$, which induces an antipodal map $\mathbf{Hom}(K_2, G) \to \mathbf{Hom}(K_2, K_{k+2})$. As we have seen in Example 4.6, there is an antipodal map $\mathbf{Hom}(K_2, K_{k+2}) \to S^k$. Composing these maps, we get an antipodal map $S^{k+1} \to S^k$, contradicting the Borsuk-Ulam Theorem. □

The following related theorem shows why using the homomorphism complex leads to generalizations.

Theorem 4.10 [4]. *If $\mathbf{Hom}(C_{2r+1}, G)$ is k-connected as a topological space for some $r \geq 1$, then the chromatic number of G is at least $k + 4$.*

The *Kneser graph* K_k^n is defined as the graph whose nodes are all k-subsets of an n-set, and two are adjacent if and only if they are disjoint ($n \geq k$).

Theorem 4.11 [32] *For $n \geq 2k$, the chromatic number of the Kneser graph K_k^n is $n-2k+2$.*

Corollary 4.12 *If all k-element subsets of a $(2k + r - 1)$-element set are divided into r classes, then one of the classes contains two disjoint k-sets.*

By Theorem 4.9, it suffices to prove the following:

Lemma 4.13 *The $\mathcal{N}(K_k^n)$ is $(n - 2k - 1)$-connected.*

Proof. The neighborhood complex $\mathcal{N}(K_k^n)$ can be described as follows: its vertices are all k-subsets of $V = \{1, \ldots, n\}$, and vertices $S_1, \ldots, S_m\}$ form a simplex if any only if $|S_1 \cup \cdots \cup S_m| \leq n - k$. It was shown in Example 2.15 that this complex is $(n - 2k - 1)$-connected. □

4.3 The Ham Sandwich Theorem

Theorem 4.14 *Let $A_1, \ldots, A_d$ be measurable sets in $\mathbb{R}^d$ with finite measure. Then there exists a closed halfspace H such that $\mu(H \cap A_i) = \frac{1}{2}\mu(A_i)$ for all i.*

Proof. For every vector $v = (v_0, v_1, \ldots, v_d) \in S^d$, we define

$$f_i(v) = \lambda\{x \in A_i :\ v_0 + v_1x_1 + \cdots + v_dx_d \geq 0\} - \lambda\{x \in A_i :\ v_0 + v_1x_1 + \cdots + v_dx_d \leq 0\}$$

(where λ is the Lebesgue measure). The map $v \mapsto (f_1(v), \ldots, f_d(v))$ is antipodal, so by the Borsuk-Ulam Theorem there is a $v \in S^d$ such that $f_1(v) = \cdots = f_d(v) = 0$. But then the halfspace H defined by the inequality $v_0 + v_1x_1 + \cdots + v_dx_d \geq 0$ satisfies the conditions of the Theorem. (We need that the hyperplane $v_0 + v_1x_1 + \cdots + v_dx_d = 0$ has measure 0.) □

4.4 The Necklace problem

Theorem 4.15 *Consider an (open) necklace consisting of pearls of k colors. Suppose that there is an even number of pearls of each color. Then we can split the necklace at k points, and divide the arising pieces between two robbers so that each robber gets exactly half of the pearls of each color.*

Theorem 4.16 (Continuous version) *Let* $A_1 \cup \cdots \cup A_k$ *be a partition of* $[0,1]$ *into measurable parts. Then there is a partition* $J_1 \cup \cdots \cup J_{k+1}$ *of* $[0,1]$ *into intervals, and a set* $I \subseteq \{1,\dots,k+1\}$ *such that*

$$\sum_{r\in I} \lambda(J_r \cap A_i) = \frac{1}{2}\lambda(A_i)$$

for every $i = 1,\dots,k$.

Proof. For $z \in S^k$, let $J_1(z) \cup \cdots \cup J_{k+1}(z)$ be the partition of $[0,1]$ into intervals with $\lambda(J_i(z)) = z_i^2$. Define

$$f_i(z) = \sum_{r=1}^{k+1} \operatorname{sgn}(z_r)\lambda(A_i \cap J_r(z)) \quad (i = 1,\dots,k).$$

Then f_i is continuous (this needs a little argument), and $f_i(-z) = -f_i(z)$. Hence by the Borsuk-Ulam Theorem, there is a $z \in S^k$ such that $f_1(z) = \cdots = f_k(z) = 0$. Then $I = \{j : \operatorname{sgn}(z_r) = 1\}$ satisfies the requirements of the theorem. □

References

[1] N. Alon, Splitting necklaces, *Adv. in Math.* **63** (1987), 247–253.

[2] N. Alon, I. Bárány, Z. Füredi, D. Kleitman, *Point selections and weak ϵ-nets for convex hulls*

[3] N. Alon, P. Frankl, L. Lovász, *The chromatic number of Kneser hypergraphs*, Trans. Amer. Math. Soc. **298** (1986), 359–370.

[4] E. Babson and D.N. Kozlov: Proof of the Lovász conjecture, *Annals of Math.* **165** (2007), 965–1007.

[5] R. Bacher, Y. Colin de Verdière, *Multiplicités des valeurs propres et transformations étoile-triangle des graphes*, preprint, 1994.

[6] E.G. Bajmóczy, I. Bárány, On a common generalization of Borsuk's and Radon's theorem, *Acta Mathematica Academiae Scientiarum Hungaricae* 34 (1979) 347–350.

[7] I. Bárány, Z. Füredi, L. Lovász, On the number of halving planes *Combinatorica* **10**, 175–183.

[8] I. Bárány, S.B. Shlosman and A. Szűcs, On a topological generalization of a theorem of Tverberg, *J. London Math. Soc.* **23**, 158–164.

[9] M. Ben-Or: Lower bounds for algebraic computation trees, *15th ACM STOC* (1983) 80–86.

[10] A. Björner, Topological Methods, in: *Handbook of Combinatorics* (ed. R. Graham, M. Grötschel, L. Lovász), Chapter 34, 1819–1872. North-Holland, Amsterdam (1995).

[11] A. Björner, Some Cohen-Macaulay Complexes arising in Group Theory, in *Commutative Algebra and Combinatorics* (ed. M. Nagata and H. Matsumura), Advanced Studies in Pure Mathematics **11**, North-Holland, Amsterdam, 1987, pp. 13–19.

[12] A. Björner, B. Korte, L. Lovász, *Homotopy properties of greedoids*, Advances in Appl. Math. **6** (1985), 447–494.

[13] A. Björner, L. Lovász, A. Yao, Linear decision trees, hyperplane arrangements, and Möbius functions, in: *Proc. 24th ACM SIGACT Symp. on Theory of Computing*, 170–177.

[14] A. Björner, L. Lovász, Linear decision trees, subspace arrangements, and Möbius functions *Journal of the Amer. Math. Soc.* **7**, 677–706.

[15] The connectivity of chessboard complexes (A. Björner, S. Vrec̆ica, R. Živaljevic̆) *Journal of the London Math Soc.* **49** (1994) 25–39.

[16] T. Böhme, On spatial representations of graphs, in: *Contemporary Methods in Graph Theory* (R. Bodendieck, ed.), BI-Wiss.-Verl. Mannheim, Wien/Zurich, 1990, pp. 151–167.

[17] Y. Colin de Verdière, Sur un nouvel invariant des graphes et un critère de planarité, *Journal of Combinatorial Theory, Series B* 50 (1990) 11–21.

[18] Y. Colin de Verdière, On a new graph invariant and a criterion for planarity, in: *Graph Structure Theory* (N. Robertson, P. Seymour, eds.), Contemporary Mathematics, American Mathematical Society, Providence, Rhode Island, 1993, pp. 137–147.

[19] A. Dold: *Lectures on Algebraic Topology*, Springer-Verlag, 1972.

[20] A. Dold: Parametrized Borsuk–Ulam theorems, *Comment. Math. Helv.* **63** (1988) 275–285.

[21] J. Folkman, The homology groups of a lattice, *J. Math. Mech.* **15** (1966) 631–636.

[22] M. Goresky and R. MacPherson: *Stratified Morse Theory*, Ergebnisse, Band 14, Springer-Verlag, Berlin (1988)

[23] E. Győri, On division of graphs to connected subgraphs, in: *Combinatorics I* (eds. A. Hajnal and V.T. Sós), Coll. Math. Soc. J. Bolyai **18** (1987) 485–494.

[24] H. van der Holst, A short proof of the planarity characterization of Colin de Verdière, *Journal of Combinatorial Theory, Series B* 65 (1995) 269–272.

[25] H. van der Holst, M. Laurent, A. Schrijver, On a minor-monotone graph invariant, *Journal of Combinatorial Theory, Series B* 65 (1995) 291–304.

[26] H. van der Holst, L. Lovász, A. Schrijver, On the invariance of Colin de Verdière's graph parameter under clique sums, *Linear Algebra and Its Applications* 226 (1995) 509–517.

[27] J. Kahn, M. Saks and D. Sturtevant, A topological approach to evasiveness, it Combinatorica **4** (1984) 297–306.

[28] D. Kozlov: *Combinatorial Algebraic Topology*, Springer, Berlin, 2008.

[29] M.A. Krasnoselsky and P.P. Zabrejko, *Geometricheskie Zadachi Nelinejnogo Analiza*, Nauka, Moscow, 1975.

[30] L. Lovász, On the number of halving lines, *Annales Univ. Eötvös* **14** (1971), 107-108.

[31] L. Lovász, A homology theory for spanning trees of a graph, *Acta Math. Hung.* **30**, 241-251.

[32] L. Lovász, Kneser's conjecture, chromatic number, and homotopy, *J. Comb. Theory A* **25** (1978), 319-324.

[33] L. Lovász, Topological and algebraic methods in graph theory, in: *Graph Theory and Related Topics*, Academic Press, 1979, 1-14.

[34] L. Lovász, Matroids and Sperner's Lemma, *Europ. J. Combin.* **1**, 65-66.

[35] L. Lovász, Self-dual polytopes and the chromatic number of distance graphs on the sphere, *Acta Sci. Math. Szeged* **45**, 317-323.

[36] L. Lovász, A. Schrijver, A Borsuk theorem for antipodal links and a spectral characterization of linklessly embeddable graphs (preprint)

[37] J. Mather: Invariance of the homology of a lattice, *Proc. Amer. Math. Soc.* **17** (1966) 1120–1124.

[38] J. Matoušek: *Using the Borsuk-Ulam Theorem: Lectures on Topological Methods in Combinatorics and Geometry*, Springer (2003).

[39] J. Milnor: On the Betti numbers of real algebraic varieties, *Proc. Amer. Math. Soc.* **15** (1964) 275–280.

[40] J.R. Munkres: *Elements of Algebraic Topology*, Addison-Wesley, Menlo Park (1984).

[41] R. Oliver: Fixed-point sets of group actions on finite acyclic complexes, *Comment. Math. Helvet.* **50** (1975), 155–177.

[42] N. Robertson, P.D. Seymour, R. Thomas, A survey of linkless embeddings, in: *Graph Structure Theory* (N. Robertson, P. Seymour, eds.), Contemporary Mathematics, American Mathematical Society, Providence, Rhode Island, 1993, pp. 125–136.

[43] N. Robertson, P. Seymour, R. Thomas, Sachs' linkless embedding conjecture, *Journal of Combinatorial Theory, Series B* 64 (1995) 185–227.

[44] R.P. Stanley: *Enumerative Combinatorics*, Vol. 1, Wadsworth & Brooks/Cole, Monterey (1986).

[45] M. Steele and A. Yao, Lower bounds for algebraic decision trees, *J. Algorithms* **3** (1982), 1–8.

[46] R. Thom: Sur l'homologie des variétés algébraiques réelles, in: *Differential and Algebraic Topology* (ed. S. S. Cairns), Princeton Univ. Press, Princeton (1965).

[47] K. Wagner, Über eine Eigenschaft der ebene Komplexe, *Mathematische Annalen* 114 (1937) 570–590.

[48] R. Živaljević: User's guide to equivariant methods in combinatorics, *Publ Inst. Math. Belgrade* **59(73)** (1996), 114–130.

[49] R.T. Živaljević, S.T. Vrećica: The colored Tverberg's problem and complexes of injective functions, *J. Combinatorial Theory* Ser. A **61** (1991) 309–318.